Solving by Factorising

Fact — If $pq = 0$ then $p = 0$ or $q = 0$. This is why we factorise: it is the *only* reason a factorised quadratic gives its solutions. (Note $pq = 6$ tells you nothing of the kind.)

Example

Solve:

1. $2x^2 + 3x = 0$
2. $12x^2 - 14x = 6$
3. $4x^2 - 9 = 0$

1. $x(2x+3) = 0 \implies x = 0$ *or* $x = -\frac{3}{2}$ *(do not divide through by x — it deletes a solution)*
2. $6x^2 - 7x - 3 = 0 \implies (3x+1)(2x-3) = 0 \implies x = -\frac{1}{3}$ *or* $\frac{3}{2}$
3. $(2x-3)(2x+3) = 0 \implies x = \pm\frac{3}{2}$

Quadratics in Disguise

Example

Solve:

1. $x^4 - 13x^2 + 36 = 0$
2. $x - \sqrt{x} - 6 = 0$
3. $2^{2x} - 9 \cdot 2^x + 8 = 0$

1. Let $u = x^2$: $u^2 - 13u + 36 = (u-4)(u-9) = 0$, so $x^2 = 4$ or 9: $x = \pm 2, \pm 3$.
2. Let $u = \sqrt{x}$ $(u \geq 0)$: $u^2 - u - 6 = (u-3)(u+2) = 0$, so $u = 3$ $(u = -2$ rejected): $x = 9$.
3. Let $u = 2^x$: $u^2 - 9u + 8 = (u-1)(u-8) = 0$, so $2^x = 1$ or 8: $x = 0$ or 3.

Textbook Exercises: SPS Course 2.7, Revision Exercise 2.7.0

Completing the Square

Fact —

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

For a non-monic quadratic, first take out the coefficient of x^2 from the x-terms.

Example

Write in the form $a(x+p)^2 + q$:

1. $x^2 - 6x + 1$
2. $3x^2 - 12x + 5$
3. $2 + 8x - x^2$

1. $(x-3)^2 - 8$
2. $3(x^2 - 4x) + 5 = 3\left[(x-2)^2 - 4\right] + 5 = 3(x-2)^2 - 7$
3. $-(x^2 - 8x) + 2 = -\left[(x-4)^2 - 16\right] + 2 = 18 - (x-4)^2$

Example

By completing the square, solve $2x^2 - 12x + 7 = 0$, giving exact answers.

$$2(x-3)^2 - 11 = 0 \implies (x-3)^2 = \frac{11}{2} \implies x = 3 \pm \sqrt{\frac{11}{2}} = 3 \pm \frac{\sqrt{22}}{2}$$

Textbook Exercises: SPS Course 2.9, Exercises 2 and 3; SPS Course 2.7, Exercises 1A and 1B

The Quadratic Formula

Theorem

$$ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Derive this by completing the square on $ax^2 + bx + c = 0$.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \implies \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$
$$\implies x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

Solve $2x^2 - 5x + 1 = 0$, giving your answers in exact form.

$$x = \frac{5 \pm \sqrt{25 - 8}}{4} = \frac{5 \pm \sqrt{17}}{4}$$

> **Remark.** All written steps must be shown: quote the formula, substitute, simplify. A calculator answer alone scores nothing.

Textbook Exercises: SPS Course 2.7, Exercise 2

The Discriminant

Definition. The discriminant of $ax^2 + bx + c$ is $b^2 - 4ac$ — the quantity under the square root in the formula.

$b^2 - 4ac > 0$	two distinct real roots
$b^2 - 4ac = 0$	one repeated root (the graph touches the x-axis)
$b^2 - 4ac < 0$	no real roots

Fact (Quadratic inequalities) — To solve $ax^2 + bx + c < 0$ (or > 0): find the roots, sketch the parabola, read off the interval. Never divide an inequality by a variable.

Example

Find the values of k for which $x^2 + 2kx + 25 = 0$ has real roots.

$4k^2 - 100 \geq 0 \implies k^2 \geq 25 \implies k \leq -5$ *or* $k \geq 5.$

Example

Find the values of p for which $px^2 + px - 6x + 2 = 0$ has no real roots.

$px^2 + (p-6)x + 2 = 0$: $\quad (p-6)^2 - 8p < 0$

$p^2 - 20p + 36 < 0 \implies (p-2)(p-18) < 0 \implies 2 < p < 18$

Example

The line $y = 2x + c$ is a tangent to the curve $y = x^2 - 4x + 5$. Find c.

$x^2 - 4x + 5 = 2x + c \implies x^2 - 6x + (5 - c) = 0$
Tangency = repeated root: $36 - 4(5 - c) = 0 \implies c = -4$.

Textbook Exercises: SPS Course 2.7, Exercises 3 and 4

The Vertex and Sketching

Fact — $y = a(x+p)^2 + q$ has vertex $(-p, q)$: minimum if $a > 0$, maximum if $a < 0$. The line of symmetry is $x = -p$.

A full sketch shows the shape, the vertex, the y-intercept and any x-intercepts.

Example

Sketch $y = 2x^2 - 8x + 3$, labelling the vertex and all intercepts.

$y = 2(x-2)^2 - 5$: vertex $(2, -5)$, minimum.
y-intercept $(0, 3)$. x-intercepts: $(x-2)^2 = \frac{5}{2} \implies x = 2 \pm \sqrt{\frac{5}{2}}$.

Example

$f(x) = 18 - (x-4)^2,\ x \in \mathbb{R}$.

1. State the maximum value of f and the range of f.
2. The domain is restricted to $x \geq k$ so that f^{-1} exists. State the smallest such k and find f^{-1}.

1. Maximum 18 at $x = 4$; range $y \leq 18$.
2. $k = 4$. $x = 18 - (y-4)^2 \implies f^{-1}(x) = 4 + \sqrt{18 - x}$ (positive root: range of f^{-1} is $y \geq 4$), domain $x \leq 18$.

Example

A quadratic has vertex $(3,-2)$ and passes through $(5,6)$. Find its equation.

$y = a(x-3)^2 - 2$ *with* $6 = a(4) - 2 \implies a = 2$: $\quad y = 2(x-3)^2 - 2$.

Textbook Exercises: SPS Course 2.7, Revision Exercise 5 and Exam Questions 2.7.8